Chapter 3

Making Quick Work of Quadratic Equations

In This Chapter

- Solving quadratic equations by factoring or taking roots
- Using the quadratic formula
- Coming to grips with quadratic inequalities

A *quadratic equation* contains a variable term with an exponent of 2, and no term with a higher power. The standard form is $ax^2 + bx + c = 0$. Quadratic equations potentially have two real solutions. You may not find two, but you start out assuming that you'll find two and then proceed to prove or disprove your assumption. Quadratic equations also serve as good models for practical applications.

In this chapter, you discover many ways to approach both simple and advanced quadratic equations. You can solve some quadratic equations in only one way, and you can solve others by readers' choice (factoring, quadratic formula, or by-guess-or-by-golly) — whatever your preference. It's nice to be able to choose. But if you have a choice, I hope you choose the quickest and easiest ways possible, so I cover these first in this chapter (except for by-guess-or-by-golly).

Using the Square Root Rule When Possible

Some quadratic equations are easier to solve than others. Half the battle is recognizing which equations are easy and which are more challenging.

If a quadratic equation is made up of a squared term and a number, written in the form $x^2 = k$, you solve the equation using the *square root rule:* If $x^2 = k$, $x = \pm\sqrt{k}$.

The number represented by k has to be positive if you want to find real answers with this rule. If k is negative, you get an imaginary answer, such as $3i$ or $5 - 4i$. (For more on imaginary numbers, check out Chapter 12.)

Solve for x using the square root rule: $6x^2 = 96$.

The initial equation doesn't strictly follow the format for the square root rule because of the coefficient 6, but you can get to the proper form pretty quickly. You divide each side of the equation by the coefficient; in this case, you get $x^2 = 16$; now you're in business. Taking the square root of each side, you get $x = \pm 4$.

Solve $y^2 = 40$ for y.

$$\begin{aligned} y^2 &= 40 \\ y &= \pm\sqrt{40} \\ &= \pm\sqrt{4}\sqrt{10} \\ &= \pm 2\sqrt{10} \end{aligned}$$

You can use a law of radicals to simplify a radical expression. Separate the number under the radical into two factors — one of which is a perfect square: $\sqrt{a \cdot b} = \sqrt{a}\sqrt{b}$.

Solving Quadratic Equations by Factoring

When you can *factor* a quadratic expression that's part of a quadratic equation you can solve quadratic equations by setting the factored expression equal to zero (making it an equation) and then using the multiplication property of zero (MPZ; see Chapter 1). How you factor the expression depends on the number of terms in the quadratic and how those terms are related.

Factoring quadratic binomials

You can factor a quadratic binomial in one of two ways — if you can factor it at all (you may find no common factor, or the two terms may not both be squares):

- Divide out a common factor from each of the terms.
- Write the quadratic as the product of two binomials, if the quadratic is the difference of perfect squares.

Taking out a greatest common factor

The *greatest common factor* (GCF) of two or more terms is the largest number (and variable combination) that divides each of the terms evenly.

EXAMPLE

Solve the equation $4x^2 + 8x = 0$ using factorization and the MPZ.

Factor out the GCF: $4x(x + 2) = 0$.

Using the MPZ, you can now make one of three statements about this equation:

- $4 = 0$, which is false — this isn't a solution
- $x = 0$
- $x + 2 = 0$, which means that $x = -2$

You find two solutions for the original equation $4x^2 + 8x = 0$: $x = 0$ or $x = -2$. If you replace the x's with either of these solutions, you create a true statement.

WARNING!

Be careful when the GCF of an expression is just x, and always remember to set that front factor, x, equal to 0 so you don't lose one of your solutions. A really common error in algebra is to take a perfectly nice equation such as $x^2 + 5x = 0$, factor it into $x(x + 5) = 0$, and give the answer $x = -5$. Don't forget the solution $x = 0$!

Factoring the difference of squares

Use the factorization of the difference of squares to solve some quadratic equations.

This method states that if $x^2 - a^2 = 0$, $(x - a)(x + a) = 0$, and $x = a$ or $x = -a$. Generally, if $k^2x^2 - a^2 = 0$, $(kx - a)(kx + a) = 0$, and $x = \frac{a}{k}$ or $x = -\frac{a}{k}$.

EXAMPLE

Solve $49y^2 - 64 = 0$ using factorization and the MPZ.

Factor the terms on the left: $(7y - 8)(7y + 8) = 0$.

And using the MPZ, $y = \frac{8}{7}$ or $y = -\frac{8}{7}$.

Factoring quadratic trinomials

Like quadratic binomials, a quadratic trinomial can have as many as two solutions — or it may have one solution or no solution at all. If you can factor the trinomial and use the MPZ to solve for the roots, you're home free. If the trinomial doesn't factor, or if you can't figure out how to factor it, you can utilize the *quadratic formula* (see the section "The Quadratic Formula to the Rescue," later in this chapter). The rest of this section deals with the trinomials that you *can* factor.

EXAMPLE

Solve $x^2 - 2x - 15 = 0$ for x.

You can factor the left side of the equation into $(x - 5)(x + 3) = 0$ and then set each factor equal to 0. When $x - 5 = 0$, $x = 5$, and when $x + 3 = 0$, $x = -3$.

EXAMPLE

Solve $24x^2 + 52x - 112 = 0$ for x.

It may not be immediately apparent how you should factor such a seemingly complicated trinomial. But factoring 4 out of each term to simplify the picture a bit, you get $4(6x^2 + 13x - 28) = 0$.

Factoring the quadratic in the parentheses: $4(3x - 4)(2x + 7) = 0$. Setting the two binomials equal to 0, you get $x = \frac{4}{3}$ or $x = -\frac{7}{2}$.

The Quadratic Formula to the Rescue

The quadratic formula is a wonderful tool to use when other factoring methods fail (see the previous section). You take the numbers from a quadratic equation, plug them into the formula, and out come the solutions of the equation. You can even use the formula when the equation does factor, but you don't see how.

The *quadratic formula* states that when you have a quadratic equation in the form $ax^2 + bx + c = 0$ (where $a \neq 0$), the equation has the solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Realizing rational solutions

You can factor quadratic equations such as $48x^2 - 155x + 125 = 0$ to find their solutions, but the factorization may not leap right out at you when the numbers are so large. Using the quadratic formula for this example, you let $a = 48$, $b = -155$, and $c = 125$. Filling in the values and solving for x, you get

$$\begin{aligned} x &= \frac{-(-155) \pm \sqrt{(-155)^2 - 4(48)(125)}}{2(48)} \\ &= \frac{155 \pm \sqrt{24{,}025 - 24{,}000}}{96} \\ &= \frac{155 \pm \sqrt{25}}{96} \\ &= \frac{155 \pm 5}{96} \end{aligned}$$

Starting with the plus sign, the first solution is $\frac{155+5}{96} = \frac{160}{96} = \frac{5}{3}$. For the minus sign, you get $\frac{155-5}{96} = \frac{150}{96} = \frac{25}{16}$. The fact that you get fractions tells you that you could've factored the quadratic: $48x^2 - 155x + 125 = (3x - 5)(16x - 25) = 0$. Do you see where the 3 and 5 and the 16 and 25 come from in the answers?

Investigating irrational solutions

The quadratic formula is especially valuable for solving quadratic equations that don't factor. Unfactorable equations, when they do have real solutions, have irrational numbers in their answers. *Irrational numbers* have no fractional equivalent; they feature decimal values that go on forever and never have patterns that repeat.

EXAMPLE

Solve the quadratic equation $2x^2 + 5x - 6 = 0$.

Using the quadratic formula, you let $a = 2$, $b = 5$, and $c = -6$, to get the following:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(2)(-6)}}{2(2)} \\ &= \frac{-5 \pm \sqrt{25 + 48}}{4} \\ &= \frac{-5 \pm \sqrt{73}}{4} \end{aligned}$$

The answer $\frac{-5 + \sqrt{73}}{4}$ is approximately 0.886, and $\frac{-5 - \sqrt{73}}{4}$ is approximately –3.386. You find perfectly good answers, rounded off to the nearest thousandth. The fact that the number under the radical isn't a perfect square tells you something else: You couldn't have factored the quadratic, no matter how hard you tried.

Promoting Quadratic-like Equations

A *quadratic-like trinomial* is a trinomial of the form $ax^{2n} + bx^n + c = 0$. The power on one variable term is twice that of the other variable term, and a constant term completes the picture. The good thing about quadratic-like trinomials is that they're candidates for factoring and then for the application of the MPZ. Solve these equations by factoring the trinomials into the product of binomials and then applying the MPZ.

Solve: $z^6 - 26z^3 - 27 = 0$.

You can think of this equation as being like the quadratic $x^2 - 26x - 27$, which factors into $(x - 27)(x + 1)$. If you replace the x's in the factorization with z^3, you have the factorization for the equation with the z's:

$$z^6 - 26z^3 - 27 = (z^3 - 27)(z^3 + 1) = 0$$

Setting each factor equal to 0:

$$\begin{aligned} z^3 - 27 &= 0 \\ z^3 &= 27 \\ z &= 3 \end{aligned} \qquad\qquad \begin{aligned} z^3 + 1 &= 0 \\ z^3 &= -1 \\ z &= -1 \end{aligned}$$

Solve the quadratic-like trinomial $y^4 - 17y^2 + 16 = 0$.

Factor the trinomial into the product of two binomials. Then factor each binomial using the rule for the difference of squares:

$$\begin{aligned} y^4 - 17y^2 + 16 &= \left(y^2 - 16\right)\left(y^2 - 1\right) \\ &= \left(y - 4\right)\left(y + 4\right)\left(y - 1\right)\left(y + 1\right) \end{aligned}$$

Setting the individual factors equal to 0, you get $y = 4$, $y = -4$, $y = 1$, $y = -1$.

Solving Quadratic Inequalities

A *quadratic inequality* is just what it says: an inequality ($<$, $>$, $\leq$, or $\geq$) that involves a quadratic expression. You can employ the same method you use to solve a quadratic inequality to solve high-degree inequalities and rational inequalities (which contain variables in fractions).

You need to be able to solve quadratic equations in order to solve quadratic inequalities. With quadratic equations, you set the expressions equal to 0; inequalities use the same numbers that give you zeros and then determine what's on either side of the numbers (positives and negatives).

To solve a quadratic inequality, follow these steps:

1. **Move all the terms to one side of the inequality sign.**
2. **Factor, if possible.**
3. **Determine all zeros (roots or solutions).**

   Zeros are the values of x that make each factored expression equal to 0.
4. **Put the zeros in order on a number line.**
5. **Create a sign line to show where the expression in the inequality is positive or negative.**

   A *sign line* shows the signs of the different factors in each interval. If the expression is factored, show the signs of the individual factors.
6. **Determine the solution, writing it in inequality notation or interval notation (see Chapter 2).**

Keeping it strictly quadratic

The techniques you use to solve the inequalities in this section are also applicable for solving higher-degree polynomial inequalities and rational inequalities. If you can factor a third- or fourth-degree polynomial (see "Promoting Quadratic-like Equations" to get started), you can handily solve an inequality where the polynomial is set less than 0 or greater than 0. You can also use the sign-line method to look at factors of rational (fractional) expressions. For now, however, consider sticking to the quadratic inequalities.

To solve the inequality $x^2 - x > 12$, for example, you need to determine what values of x you can square so that when you subtract the original number, your answer will be bigger than 12. For example, when $x = 5$, you get $25 - 5 = 20$. That's certainly bigger than 12, so the number 5 works; $x = 5$ is a solution. How about the number 2? When $x = 2$, you get $4 - 2 = 2$, which isn't bigger than 12. You can't use $x = 2$ in the solution. Do you then conclude that smaller numbers don't work? Not so. When you try $x = -10$, you get $100 + 10 = 110$, which is most definitely bigger than 12. You can actually find infinitely many numbers that make this inequality a true statement.

Therefore, you need to solve the inequality by using the steps I outline in the introduction to this section:

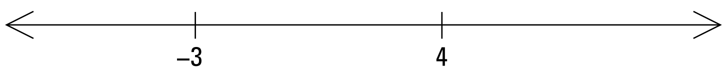
1. **Subtract 12 from each side of the inequality $x^2 - x > 12$ to move all the terms to one side.**

   You end up with $x^2 - x - 12 > 0$.

2. **Factoring on the left side of the inequality, you get $(x - 4)(x + 3) > 0$.**

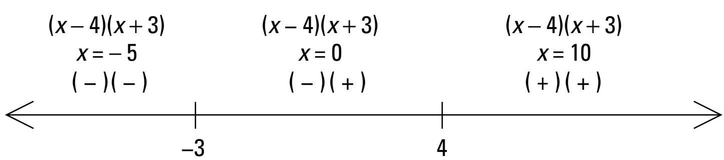
3. **Determine that all the zeroes for the inequality are $x = 4$ and $x = -3$.**

4. **Put the zeros in order on a number line, shown in the following figure.**



5. **Create a sign line to show the signs of the different factors in each interval.**

   Between –3 and 4, try letting $x = 0$ (you can use any number between –3 and 4). When $x = 0$, the factor $(x - 4)$ is negative, and the factor $(x + 3)$ is positive. Put those signs on the sign line to correspond to the factors. Do the same for the interval of numbers to the left of –3 and to the right of 4 (see the following figure).



   REMEMBER: The x values in each interval are really random choices (as you can see from my choice of $x = -5$ and $x = 10$). Any number in each of the intervals gives you the same positive or negative value to the factor.

6. **To determine the solution, look at the signs of the factors; you want the expression to be positive, corresponding to the inequality *greater than zero.***

The interval to the left of –3 has a negative times a negative, which is positive. So, any number to the left of –3 works. You can write that part of the solution as $x < -3$ or, in interval notation, $(-\infty, -3)$. The interval to the right of 4 has a positive times a positive, which is positive. So, $x > 4$ is a solution; you can write it as $(4, \infty)$. No matter what numbers you choose in the interval between –3 and 4, the result is always negative because you have a negative times a positive. The complete solution lists both intervals that have working values in the inequality.

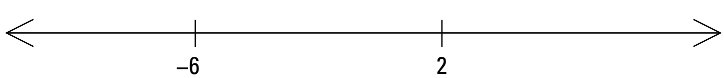
The solution of the inequality $x^2 - x > 12$, therefore, is $x < -3$ or $x > 4$.

Signing up for fractions

The sign-line process (see the introduction to this section and the previous example problem) is great for solving rational inequalities, such as $\frac{x-2}{x+6} \le 0$. The signs of the results of multiplication and division use the same rules, so to determine your answer, you can treat the numerator and denominator the same way you treat two different factors in multiplication.

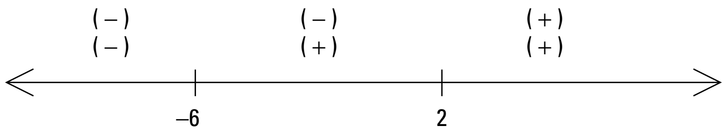
Using the steps from the list I present in the introduction to this section, determine the solution for a rational inequality:

1. **Every term in $\frac{x-2}{x+6} \le 0$ is to the left of the inequality sign.**
2. **Neither the numerator nor the denominator factors any further.**
3. **The two zeros are $x = 2$ and $x = -6$.**
4. **You can see the two numbers on a number line in the following illustration.**



5. **Create a sign line for the two zeroes.**

   You can see in the following figure that the numerator is positive when x is greater than 2, and the denominator is positive when x is greater than –6.



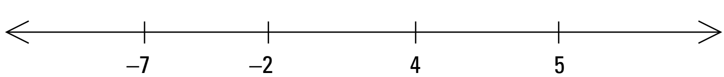
6. **When determining the solution, keep in mind that the inequality calls for something less than or equal to zero.**

   The fraction is a negative number when you choose an x between –6 and 2. You get a negative numerator and a positive denominator, which gives a negative result. Another solution to the original inequality is the number 2. Letting $x = 2$, you get a numerator equal to 0, which you want because the inequality is less than or equal to 0. You can't let the denominator be 0, though. Having a zero in the denominator isn't allowed because no such number exists. So, the solution of $\frac{x-2}{x+6} \le 0$ is $-6 < x \le 2$. In interval notation, you write the solution as $(-6, 2]$.

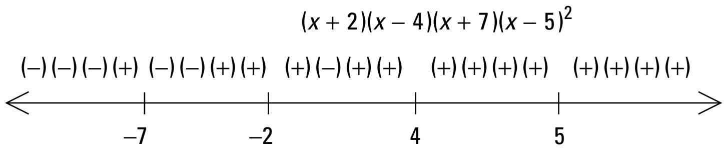
Increasing the number of factors

The method you use to solve a quadratic inequality (see the "Keeping it strictly quadratic" section, earlier in this chapter) works nicely with fractions and high-degree expressions. For example, you can solve $(x + 2)(x - 4)(x + 7)(x - 5)^2 \ge 0$ by creating a sign line and checking the products.

The inequality is already factored, so you move to the step (Step 3) where you determine the zeros. The zeros are –2, 4, –7, and 5 (the 5 is a double root and the factor is always positive or 0). The following figure shows the values in order on the number line.



Now you choose a number in each interval, substitute the numbers into the expression on the left of the inequality, and determine the signs of the four factors in those intervals. You can see from the following figure that the last factor, $(x-5)^2$, is always positive or 0, so that's an easy factor to pinpoint.



You want the expression on the left to be positive or 0, given the original language of the inequality. You find an even number of positive factors between –7 and –2 and for numbers greater than 4. You include the zeros, so the solution you find is $-7 \leq x \leq -2$ or $x \geq 4$. In interval notation, you write the solution as $[-7, -2]$ or $[4, \infty)$.